Lecture 13

More Examples on Mathematical Induction, Flawed Proofs

Examples: Mathematical Induction

Example: At a tennis tournament, every two players played against each other exactly one time. After all games were over, each player listed the names of those he defeated, and the names of those defeated by someone he defeated. Prove that at least one player listed the names of everybody else.

Solution: We will prove the statement for any n-player tournament, where n is an integer ≥ 2.

Basis Step: For $n = 2$, the statement is trivially true because winner of the only match will list the name of the loser.

Inductive Step: Assume that the statement is true for a k-players tournament.

Under this assumption we will prove the statement for $(k+1)$-players tournament.

Let A be one of the players with least number of victories in a $(k+1)$-players tournament.

continue…

Examples: Mathematical Induction

Let's temporarily take out A from the $(k+1)$-player tournament.

From the induction hypothesis, in the remaining k player, there will be one, say B, who has listed the name of rest of the $(k-1)$ players.

"Put A back" in the tournament and consider the two cases:

$1)$ Either B defeated A or B defeated someone who defeated A:

In this case, A will feature in B's list making it a list of k players, and we are done.

$2)$ Neither B defeated A nor B defeated someone who defeated A:

That means A defeated B and all the players defeated by B.

But this implies that A won more games than B, a contradiction.

So, Case 2 is not possible. $\blacksquare$

Examples: Mathematical Induction

Example: Prove that $3^n > n^4$, for $n \geq 8$. (Notice the domain is not $\mathbb{Z}^+$.)

Solution:

Basis Step: For $n = 8$, $3^8 = 6561 > 8^4 = 4096$.

Inductive Step: For any $k \geq 8$, we now assume that $3^k > k^4$.

And under this assumption we prove that $3^{k+1} > (k+1)^4$.

Take IH and multiply by 3 on both the sides.

$$3^k.3 > k^4.3$$

$$\implies \quad 3^{k+1} > 3k^4$$

If we can prove $3k^4 > (k+1)^4$ for $k \geq 8$, then we are done.

continue…

Examples: Mathematical Induction

For $k \geq 8$,

$$3k^4 > (k+1)^4 \iff \left(\frac{k}{k+1}\right)^4 > \frac{1}{3} \iff \left(1 - \frac{1}{k+1}\right)^4 > \frac{1}{3}$$

For $k = 8$,

$$\left(1 - \frac{1}{k+1}\right)^4 = (8/9)^4 = 0.624 > \frac{1}{3}$$

With growing k, $\left(1 - \frac{1}{k+1}\right)^4$ only grows, thus $\left(1 - \frac{1}{k+1}\right)^4 > \frac{1}{3}$ for $k \geq 8$.

$\blacksquare$

Incorrect usage of Mathematical Induction

False Theorem: All positive integers of the form $2n + 1$ are divisible by 2.

Incorrect Proof: We will prove the statement using mathematical induction.

Basis Step: Statement is “trivially” true for $n = 1$. ← *Error of this proof.*

Inductive Step: Assume that the statement is true for k, i.e,

$$2k + 1 \text{ is divisible by } 2$$

Under that assumption prove that it is true for $k + 1$ as well, i.e.,

$$2k + 3 \text{ is divisible by } 2$$

If $2k + 1$ is divisible by 2, then for some integer c,

$$2k + 1 = 2c$$

continue…

Incorrect usage of Mathematical Induction

$$2k + 1 + 2 = 2c + 2 \qquad \textit{(Adding } 2 \textit{ on both the sides)}$$

$$2k + 3 = 2(c + 1)$$

Thus, $2k + 3$ is divisible by 2. $\blacksquare$

Tip: Wrong proof of the base case can lead to wrong conclusion. So, be careful while proving base case.

Incorrect usage of Mathematical Induction

False Theorem: All horses are of the same colour.

Incorrect Proof: Let's rephrase the theorem as the following:

For any positive integer n, any n horses always have the same colour.

Basis Step: For $n = 1$, the statement is obviously true.

Inductive Step: We assume the statement is true for k horses and prove it for $k + 1$ horses.

Take $k + 1$ horses and line them up:

2nd, 3rd, …, $(k+1)$th horses are of same colour.

$$1 \quad 2 \quad 3 \quad \dots \quad i \quad \dots \quad j \quad \dots \quad (k-1) \quad k \quad (k+1)$$

1st, 2nd, …, kth horses are of same colour.

continue…

Incorrect usage of Mathematical Induction

Since 1st, 2nd, $\ldots$, kth horses have the same colour, and 2nd and $(k+1)$th horses have the same colours as well, all $k+1$ horses have the same colour.

$\blacksquare$

Flaw: Any k horses having same colour implies $k+1$ horses have same colour, for all positive integer values of k, except for $k = 1$.

Principle of Mathematical Induction:

To prove $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we perform two steps:

Basis Step: Prove that $P(1)$ is true.

Inductive Step: We prove that $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

The previous proof isn't doing the inductive step thoroughly.